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Note

A special k-coloring for a connected k-chromatic graph

Guantao Chen[a,*,1], Richard H. Schelp[b,2], Warren E. Shreve[c]

[a]Georgia State University, Atlanta, GA 30303, USA
[b]Memphis State University, Memphis, TN 38152, USA
[c]North Dakota State University, Fargo, ND 58105, USA

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Abstract

For each positive integer k we consider the smallest positive integer $f(k)$ (dependent only on k) such that the following holds: Each connected graph G with chromatic number $\chi(G) = k$ can be properly vertex colored by k colors so that for each pair of vertices x_0 and x_p in any color class there exist vertices $x_1, x_2, \dots, x_{p-1}$ of the same class with $\mathrm{dist}(x_i, x_{i+1}) \leqslant f(k)$ for each i, $0 \leqslant i \leqslant p-1$. Thus, the graph is k-colorable with the vertices of each color class placed throughout the graph so that no subset of the class is at a distance $> f(k)$ from the remainder of the class.

We prove that $f(k) < 12k$ when the order of the graph is $\geqslant k(k-2)+1$.

1. Introduction

Let G be a graph. For each pair (u, v) of vertices of G we let $\mathrm{dist}(u, v)$ denote the distance between u and v, and let $\chi(G)$ be the chromatic number of G. Further, if $\chi(G) = k$, then we shall consider a vertex coloring of G with colors $1, 2, \dots, k$ and let X_i denote the ith color class for each i ($1 \leqslant i \leqslant k$). Also if G is a graph and m is a fixed positive integer, then G^m will denote the mth power of G in that $V(G^m) = V(G)$ and $uv \in E(G^m)$ if $\mathrm{dist}(u, v) \leqslant m$. For $X \subseteq V(G^m)$, $G^m[X]$ will denote the subgraph induced by X in G^m. Other notation will follow that given in [1].

Graph coloring has long been a fundamental subject in graph theory. There are many papers and books which deal with various topics in graph coloring (see e.g. [2]).

* Correspondence address: Department of Mathematics and Computer Science, Georgia State University. Fargo, ND 58105-5075, USA. E-mail: gchen@plains.nodak.edu.

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In this paper, we wish to give a connected k-chromatic graph a special k-coloring – adding an additional requirement for the color classes. In fact, we show that each such graph can be colored by k colors such that no subject of vertices of any class can be separated by a great distance from the rest of the vertices in the class. Specifically, for each positive integer k we wish to find a smallest positive integer $f(k)$ such that each connected k-chromatic graph G has a k-coloring with color classes $X_1, X_2, \ldots, X_k$ with $G^{f(k)}[X_i]$ connected for all i. We will show that $f(k) \leqslant 12k - 21$ when $|V(G)| = n > k(k-2)$, $k \geqslant 2$. Also $f(k) \geqslant k$ since the graph G formed by joining two vertex-disjoint connected k-chromatic graphs by a path of length k is itself k-chromatic and has $G^{k-1}[X_i]$ disconnected for some i, where $X_1, X_2, \ldots, X_k$ are color classes of G under any k-coloring. In fact, we make the following conjecture.

Conjecture 1. Let G be a connected k-chromatic graph of order n. For sufficiently large n (dependent only on k), there exists a proper k-coloring of G with color classes $X_1, X_2, \ldots, X_k$ such that $G^k[X_i]$ is connected for all i, $1 \leqslant i \leqslant k$.

Although we have not proved the conjecture, we have found an upper bound on $f(k)$ which is best possible up to a multiplicative constant. The following theorem proves that

$$f(k) \leqslant \max\{12k - 21,\ k(k-2) + 1\}.$$

Theorem 1. (Main theorem). *If G is a connected graph with chromatic number $\chi(G) = k \geqslant 2$ on $n > k(k-2)$ vertices, then there is a proper vertex coloring of G by k colors with the color classes $X_1, X_2, \ldots, X_k$ such that $G^{12k-21}[X_i]$ is connected for each* $(1 \leqslant i \leqslant k)$.

We include a second theorem which gives a much weaker exponential bound on $f(k)$ for large k than the linear bound of Theorem 1, but does provide a universal bound for all n and improves Theorem 1 when $k \leqslant 6$.

Theorem 2. *If G is a connected graph with chromatic number $\chi(G) = k$, then there is a proper vertex coloring of G by k colors with color classes $X_1, X_2, \ldots, X_k$ such that $G^{2^{k-1}}[X_i]$ is connected for all i, $1 \leqslant i \leqslant k$.*

The formal proofs of these two results appear in the next section.

2. Proofs

Before presenting the formal proof of Theorem 1, we briefly outline the idea for the casual reader. We first color G by k colors with color classes $X_1, X_2, \ldots, X_k$ such that $|X_1| \leqslant |X_2| \leqslant \cdots \leqslant |X_k|$ and $|X_k|$ is as large as possible. Then, $|X_k| \geqslant k-1$ if

$|V(G)| > k(l-2)$. Since we assume that $|X_k|$ is as large as possible, all vertices not in X_k have at least one neighbor in X_k. Using this property, we can verify that $G^3[X_k]$ is connected. We will enlarge these color classes X_i $(1 \leqslant i \leqslant k-1)$ for which $G^{12k-21}[X_i]$ are not connected by recoloring some selected vertices in X_k in the color i such that the resulting color classes satisfy the conclusion of the theorem. Let $Y_1, Y_2, \ldots, Y_k$ denote the new color classes. We will show that $G^{12k-21}[Y_i]$ is connected for each i $(1 \leqslant i \leqslant k)$. We will select the vertices in X_k to be recolored very carefully such that the remaining vertices with the color k (Y_k) are not separated too much. In fact, we will show that $G^{6k-9}[Y_k]$ is connected which is a little better than we claimed.

The recolored vertices of X_k are selected so that if $u, v \in Y_l$, $1 \leqslant l \leqslant k-1$, then there exist vertices $x = x_0, x_1, x_2, \ldots, x_p = y \in X_k$ and vertices (now with color l) $y_0, y_1, \ldots, y_p$ of Y_l (x_i may be the same as y_j for some i and j) such that $\mathrm{dist}(x_i, x_{i+1}) \leqslant 3$ and $\mathrm{dist}(x_i, y_i) \leqslant 6(k-2)$ for all i, either $u = x$ or $ux \in E(G)$, and either $v = y$ or $vy \in E(G)$. This forces $\mathrm{dist}(y_i, y_{i+1}) \leqslant 12k-21$ for all i, and $\mathrm{dist}(u, y_0), \mathrm{dist}(y_p, v) \leqslant 12k-21$, so that u and v are connected by a path in $G^{12k-21}[Y_l]$. Also the recolored vertices are selected so no vertex of X_k is recolored more than once, and for each vertex x in X_k there is a vertex y in X_k such that vertex y is not recolored and $\mathrm{dist}(x, y) \leqslant 3k-6$. This will give that $G^{6k-9}[|Y_k]$ is connected and will complete the proof.

2.1. *Proof of Theorem 1*

Let G be a graph of $n > k(k-2)$ vertices and let $\chi(G) = k \geqslant 2$. Let $X_1, X_2, \ldots, X_k$ be a proper coloring of the vertices of G such that $|X_1| \leqslant |X_2| \leqslant \cdots \leqslant |X_k|$ with $|X_k|$ as large as possible. Clearly, $|X_k| \geqslant k-1$ and $G^3[X_k]$ is connected. In the following, we will use heavily the property that $G^3[X_k]$ is connected and establish an algorithm to recolor some vertices of X_k to obtain a proper coloring $Y_1, Y_2, \ldots, Y_k$ of vertices of G such that $G^{12k-21}[Y_i]$ is connected for $1 \leqslant i \leqslant k-1$ and such that $G^{6k-9}[Y_k]$ is connected.

Claim 1. *For every $x \in X_k$ and every positive integer $t \leqslant k-2$ there is a set $W_t[x]$ of $t+1$ vertices such that*

- $x \in W_t[x] \subseteq X_k \cap N_{3t}[x]$, *where* $N_{3t}[x] = \{v \in V(G)\colon \mathrm{dist}(v, x) \leqslant 3t\}$,
- $\mathrm{dist}(y, z) \leqslant 3t$ *for every pair of vertices y and z of $W_t[x]$.*

Proof. We prove the result inductively on t. Clearly, $W_0[x] = \{x\}$, satisfying the properties listed above. Suppose that $W_{t-1} = \{x, x_1, \ldots, x_{t-1}\}$ and it satisfies the above properties for $t-1$. Since $|X_k| \geqslant k-1$ and $G^3(X_k)$ is connected, there is a vertex $x_t, x_t \notin W_{t-1}[x]$, such that $\mathrm{dist}(x_t, W_{t-1}[x]) \leqslant 3$, where $\mathrm{dist}(x_t, W_{t-1}[x])$ denotes the shortest distance between the vertex x_t and vertex set $W_{t-1}[x]$. Let $W_t[x] = W_{t-1}[x] \cup \{x_t\}$. It is readily seen that the result holds. $\square$

We next recolor some vertices of X_k with colors $1, 2, \ldots, k-1$ by the following algorithm. In each step of the recoloring and for each $i < k$, call a (possible recolored) vertex $x \in X_k$ *i-bad* if no vertices in $N_{6(k-2)}[x]$ are colored with color i for each $i = 1, 2, \ldots, k-1$; otherwise, call the vertex x *i-good*. If a vertex $x \in X_k$ is i-good for every i $(1 \leqslant i \leqslant k-1)$, call it a *good* vertex. Note that since G is connected, every vertex $x \in X_k$ must be adjacent to at least one vertex with another color. Thus, there are at most $k-2$ colors i such that x is i-bad.

Recoloring algorithm.

1. Let $\prec$ be a total order of the vertices of X_k and set $C = \emptyset$.
2. Let x be the first element of X_k not in C under the order $\prec$.
3. If x is good, let $C = C \cup \{x\}$.
4. If x is $i_1, i_2, \ldots, i_m$-bad and j-good for each $j \notin \{i_1, i_2, \ldots, i_m\}$, select $m+1$ vertices from $W_{k-2}[x]$ which have not yet changed the color (we will prove $m+1$ such vertices exist after the following claim). Recolor m of these $m+1$ vertices with colors $i_1, i_2, \ldots, i_m$, respectively. Then let $C = C \cup \{x\}$.
5. If $C = X_k$, stop, otherwise go to step 2.

Claim 2. *For every $x \in X_k$ and every i $(1 \leqslant i \leqslant k)$, there is at most one vertex in $W_{k-2}[x]$ which has been recolored from k to i by the Recoloring Algorithm.*

Proof. Suppose, on the contrary, there are two vertices u and v in $W_{k-2}[x]$ that have been recolored from color k to color i for some i $(1 \leqslant i \leqslant k-1)$. Without loss of generality, assume that u was recolored before v was recolored. At the step when vertex v is recolored, there is a vertex $y \in X_k$ such that $v \in W_{k-2}[y]$, and no vertices in $N_{6(k-2)}[y]$ are colored with color i. Since both u and v are in $W_{k-2}[x]$, $\text{dist}(u, v) \leqslant 3(k-2)$ by Claim 1. Note that $v \in W_{k-2}[y]$ implies $\text{dist}(u, y) \leqslant 3(k-2)$. Combining these inequalities gives $\text{dist}(u, y) \leqslant 6(k-2)$, which contradicts that no vertices were colored with color i in $N_{6(k-2)}[y]$. $\square$

For each $x \in X_k$, since $|W_{k-2}[x]| = k-1$ and there are at most $k-2$ colors i such that x is i-bad, the $m+1$ vertices of $W_{k-1}[x]$ needed in the Recoloring Algorithm exist. Thus, the algorithm works and every vertex in X_k is a good vertex after application of the Recoloring Algorithm. For each i, $1 \leqslant i \leqslant k$, let Y_i denote the set of vertices that are colored i after applying the Recoloring Algorithm. Note that no vertex in X_k has been recolored more than once.

Claim 3. *For each l $(1 \leqslant l \leqslant k-1)$, $G^{12k-21}(Y_\ell)$ is connected.*

Proof. For any two vertices u and v in Y_ℓ, by the maximality of $|X_k|$, there are two vertices x and y (they may be same) in X_k such that either $u = x$ or $ux \in E(G)$, and either $v = y$ or $vy \in (G)$, i.e., $\mathrm{dist}(u,x) \leqslant 1$ and $\mathrm{dist}(v,y) \leqslant 1$. Since $G^3[X_k]$ is connected, there is a sequence

$$x = x_0, x_1, x_2, \ldots, x_p = y$$

of vertices of X_k such that $\mathrm{dist}(x_i, x_{i+1}) \leqslant 3$ for each $0 \leqslant i \leqslant p-1$. Note that every x_i is a good vertex after application of the Recoloring Algorithm. Thus, for each i there is a vertex $y_i \in Y_l$ such that $y_i \in N_{6(k-2)}(x_i)$ for each x_i, i.e., $\mathrm{dist}(x_i, y_i) \leqslant 6(k-2)$. Hence, we obtain a vertex sequence

$$y_0, y_1, \ldots, y_p$$

(some members of the sequence may be repeated) with $y_i \in Y_l$ and

$$\mathrm{dist}(y_i, y_{i+1}) \leqslant \mathrm{dist}(y_i, x_i) + \mathrm{dist}(x_i, x_{i+1}) + \mathrm{dist}(x_{i+1}, y_{i+1}) \leqslant 6(k-2) + 3$$
$$+ 6(k-2) + 3 + 6(k-2) \leqslant 12k - 21$$

for each i $(0 \leqslant i \leqslant p-1)$. Also note that

$$\mathrm{dist}(u, y_0) \leqslant \mathrm{dist}(u, x_0) + \mathrm{dist}(x_0, y_0) \leqslant 1 + 6(k-2),$$

$$\mathrm{dist}(v, y_p) \leqslant \mathrm{dist}(v, x_p) + \mathrm{dist}(x_p, y_p) \leqslant 1 + 6(k-2).$$

Thus, $G^{12k-21}[Y_l]$ is connected for each l $(1 \leqslant l \leqslant k-1)$. □

Claim 4. *$G^{6k-9}[Y_k]$ is connected.*

Proof. By Claim 2, for each $x \in X_k$, there are at most $k-2$ vertices in $W_{k-2}[x]$ which are recolored. Hence, there is a vertex in $W_{k-2}[x]$ which is still colored with the color k. Let u and v be two vertices in Y_k. Since $G^3[X_k]$ is connected, there is a vertex sequence

$$u = x_0, x_1, x_2, \ldots, x_p = v.$$

such that $\mathrm{dist}(x_i, x_{i+1}) \leqslant 3$ for each i $(1 \leqslant i \leqslant p-1)$. For each x_i there is a $y_i \in W_{k-2}[x_i] \cap Y_k$. Thus, a vertex sequence

$$u = y_0, y_1, y_2, \ldots, y_p = v \text{ is obtained}$$

(some members of the sequence may be repeated) such that $y_i \in Y_k$ and

$$\mathrm{dist}(y_i, y_{i+1}) \leqslant \mathrm{dist}(y_i, x_i) + \mathrm{dist}(x_i, x_{i+1}) + \mathrm{dist}(x_{i+1}, y_{i+1}) \leqslant 3(k-2) + 3$$
$$+ 3(k-2) \leqslant 6k - 9$$

for each $0 \leqslant i \leqslant p-1$. Hence, $G^{6k-9}(Y_k)$ is connected. □

Combining Claims 3 and 4 complete the proof of the theorem.

2.2. Proof of Theorem 2

We prove Theorem 2 by induction on the chromatic number k. Clearly, Theorem 2 holds for $k = 2$ since every connected bipartite graph has a unique 2-coloring of its vertices. Assume that G is a connected graph with chromatic number $\chi(G) = k + 1$. Let $V(G) = X_1 \cup X_2 \cup \cdots \cup X_{k+1}$ be a proper $(k + 1)$-coloring of $V(G)$ such that $|X_{k+1}|$ is as large as possible. Clearly, $G^3[X_{k+1}]$ is connected and $3 \leqslant 2^k$.

Let $G_1, G_2, \ldots, G_m$ be the connected components of $G - X_{k+1}$. Form a new graph T^* with m vertices $z_1, z_2, \ldots, z_m$ such that $z_i z_j \in E(T^*)$ if there is a vertex $x \in X_{k+1}$ such that $N(x) \cap V(G_i) \neq \emptyset$ and $N(x) \cap V(G_j) \neq \emptyset$. Here $N(x)$ indicates the neighborhood of x in the graph G. Since G is connected and X_{k+1} is an independent vertex set of G, T^* is connected. Let T be a spanning tree of T^*.

Note that for each $z_i z_j \in E(T)$ there are two vertices $x_i \in V(G_i)$ and $x_j \in V((G_j)$ such that x_i and x_j have a common neighbor in X_{k+1}. Next replace each vertex of z_i of T by G_i for $i = 1, 2, \ldots, m$, and each edge $z_i z_j$ by an edge $x_i x_j$ where $x_i \in V(G_i)$, $x_j \in V(G_j)$ such that x_i and x_j have a common neighbor in X_{k+1}. In this way a new graph H is obtained. It is not difficult to see that H is connected and $\chi(H) = k$.

By the induction hypothesis there is a proper vertex coloring of H,

$$V(H) = Y_1 \cup Y_2 \cup \cdots \cup Y_k$$

such that $H^{2k-1}(Y_i)$ is connected for each $i = 1, 2, \ldots, k$. Note that for every edge $uv \in E(H)$ either $uv \in E(G)$ or there is a vertex w in X_{k+1} such that $uw, vw \in E(G)$. Thus, $G^{2^k}[Y_i]$ is connected for each i, $i = 1, 2, \ldots, k$. Let $Y_{k+1} = X_{k+1}$. Recall $G^3[X_{k+1}]$ is connected so that the proof is complete. □

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